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Trace formulas for annuli

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Abstract: Assuming the completeness condition for boundaries we derive trace formulas for the annulus coefficients in 2-dimensional conformal field theory. We also derive polynomial equations that relate the annulus, Moebius and Klein bottle coefficients, and conjecture an annulus trace formula that is sensitive to the orientation of the boundaries.

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Contents

1. Introduction

To compute perturbative open string spectra using (unitary, rational) conformal field theory one needs to know the following data [1, 2]: a multiplicity matrix Z_{ij} that defines the torus partitition function, a set $\{a\}$ of allowed boundary conditions, a set of coefficients B_{ma} that describe the reflection of bulk fields at boundary a, and a set of coefficients Γ_m that decribe the behavior of bulk fields in the presence of a crosscap. Given these data one can compute all closed and open string partition functions. Of course, much more information is needed to compute correlation functions.

Three different labels where introduced here: The labels $i, j, \ldots$ refer to representations of the chiral algebra of the bulk CFT; the labels $a, b, \ldots$ indicate distinct boundary conditions, and the labels $m, n, \ldots$ correspond to those bulk fields that can appear in the transverse channel coupling to boundaries and/or crosscaps. More precisely the latter correspond to Ishibashi boundary and crosscap states that preserve the chiral algebra. The complete set of such states corresponds to the bulk fields that are paired with their charge conjugate, i.e. those for which $\mathbf{Z}_{ii^c} \neq 0$. In this paper we will consider arbitrary symmetric modular invariant matrices Z_{ij}. In particular this includes matrices with a non-trivial kernel, implying an extension of the chiral algebra. The Ishibashi states we will use are only required to preserve the original, unextended algebra, so that the boundaries and crosscaps may break part of the extended symmetry [3].

In principle, one would like to determine all allowed boundary and crosscap coefficients given a modular invariant Z_{ij}. This problem can be transformed from a problem over the real numbers to a problem in terms of bounded integers by describing it in terms of annulus, Moebius and Klein bottle coefficients. Just like the search for modular invariants this is still a difficult problem to solve in general, and indeed the goal of this paper is more modest. We merely want to formulate a set of polynomial equations and trace formulas that the solutions should satisfy. Part of these results can be derived under mild assumptions from the "completeness condition" to be discussed below, others are conjectures which we can only prove in special cases. The complete solution given in [4] for all Klein bottle choices and all simple current modular invariants serves as a useful guiding principle for the general case, as well as a non-trivial test for the conjectures.

A complication that is usually associated with extensions of the chiral algebra is the appearance of multiplicities larger than one. If a matrix element Z_{ii^c} is larger than one, the corresponding Ishibashi states are degenerate and we must introduce an additional degeneracy label $0 \leq \alpha(i) < Z_{ii^c}$ (to simplify the notation we shall omit in the rest of the paper the dependence of α on i). In the standard computation of the annulus, Moebius and Klein bottle amplitudes we cannot rely on purely representation-theoretic arguments anymore, because this gives no information regarding the overlap of states within the same degeneracy space. A general parametrization yields the following expressions[1]

$$\begin{aligned} A^i_{ab} &= \sum_{m,\alpha,\beta} S^i{}_m g^{\alpha\beta}_m B_{(m,\alpha)a} B_{(m,\beta)b} \,, \\ K^i &= \sum_{m,\alpha,\beta} S^i{}_m k^{\alpha\beta}_m \Gamma_{(m,\alpha)} \Gamma_{(m,\beta)} \,, \\ M^i{}_a &= \sum_{m,\alpha,\beta} P^i{}_m h^{\alpha\beta}_m B_{(m,\alpha)a} \Gamma_{(m,\beta)} \,, \end{aligned} \tag{1.1}$$

where S is the standard modular matrix, while $P = \sqrt{T} S T^2 S \sqrt{T}$ [5]. Note that k_m and g_m are symmetric matrices. This means in particular that they have a square root, which we can absorb in the definition of the coefficients B and Γ. In this way we can see that without loss of generality k and g may be replaced by the identity matrix. This changes h to a new matrix h'. Having done that, we may allow, in terms of the new coefficients, orthogonal rotations in degeneracy space, which do not alter the Klein bottle and the annulus:

$$B_{(m,\alpha)a} \longrightarrow \sum_\beta W^m_{\alpha\beta} B_{(m,\beta)a} \,, \qquad \Gamma_{(m,\alpha)} \longrightarrow \sum_\beta V^m_{\alpha\beta} \Gamma_{(m,\beta)} \,.$$

This changes the matrix h' to

$$h''_m = (W^m)^T h'_m V^m \,.$$

This allows us to diagonalize h''_m. The eigenvalues will be denoted λ^m_α. It is now instructive to transform to the transverse channel. Then the amplitudes are

$$\tilde{A}_{ab} = \sum_{m,\alpha} B_{(m,\alpha)a} B_{(m,\alpha)b} \mathcal{X}_m \,, \tag{1.2}$$

$$\tilde{K} = \sum_{m,\alpha} \Gamma_{(m,\alpha)} \Gamma_{(m,\alpha)} \mathcal{X}_m \,, \tag{1.3}$$

$$\tilde{M}{}_a = \sum_{m,\alpha} \lambda^\alpha_m B_{(m,\alpha)a} \Gamma_{(m,\alpha)} \hat{\mathcal{X}}_m \,, \tag{1.4}$$

[1] The degeneracy matrices first appeared in [4]. As we show here, they can be transformed to the identity, but in that basis the reflection and crosscap coefficients presented in [4] become more complicated.

where $\mathcal{X}_m$ is a character (with the usual arguments and the usual definition of $\hat{\mathcal{X}}$). In the absence of degeneracies the Moebius amplitude is the "geometric mean" of the annulus and the Klein bottle. The result (1.4) violates this geometric mean principle unless λ_m^α is just a sign. But then we can absorb it into the definition of the crosscap coefficients without changing the Klein bottle. Hence if we adopt the geometric mean principle we may from now on assume that all λ's are equal to 1.

2. Completeness

In string theory the identity character (which in our notation corresponds to the label 0) gives rise to gauge bosons. The resulting gauge groups are only identifiable with SO(N), Sp(N) or U(N) for generic N if ${A^0}_{ab}$ is an involution. This implies

$$\sum_{i,\alpha} S_{i0}\left[B_{(i,\alpha)a}B_{(i,\alpha)b}\right] = \delta_{ba^c}\,, \tag{2.1}$$

where a^c is, by definition, the boundary conjugate to a. The fact that boundary conjugation must be an involution implies that the reflection coefficients $R_{(m,\alpha),a} = \sqrt{S_{0m}}B_{(m,\alpha),a}$ are orthogonal

$$\sum_{(m,\alpha)} R_{(m,\alpha),a}R_{(m,\alpha),b^c} = \delta_{ab}\,.$$

This puts an upper limit on the number of distinct boundaries that can appear in a given theory: the number cannot exceed the number of Ishibashi states, counted according to their degeneracy $\mathbf{z}_{ii^c}$.

Just like modular invariance is a completeness condition for operators in the bulk CFT, it is natural to postulate a completeness condition for boundaries in the open string case, namely that the upper bound is saturated. This implies that R is a square matrix, and that there must exist an inverse $\hat{R}$ so that

$$\sum_a \hat{R}_{(m,\alpha),a}R_{(n,\beta),a} = \delta_{nm}\delta_{\alpha\beta}\,.$$

Multiplying by $R_{(n,\beta),b}$ and summing over (n,β) we find then that $\hat{R}_{(m,\alpha),a} = R_{(m,\alpha),a^c}$, so that

$$\sum_a {R_{(m,\alpha)}}^a R_{(n,\beta),a} = \delta_{nm}\delta_{\alpha\beta}\,. \tag{2.2}$$

Here we use raised indices to indicate boundary conjugation. From this form of the completeness condition one can straightforwardly derive another well-known expression

$$\sum_b {A_{ia}}^b {A_{jb}}^c = \sum_k {N_{ij}}^k {A_{ka}}^c\,, \tag{2.3}$$

where ${N_{ij}}^k$ are the fusion coefficients, expressed in terms of the Verlinde formula.

This completeness condition was first written down in [6],[2] where a heuristic argument for its validity was given. Two years later in [8] an algebraic derivation was presented similar to the one we used above.[3] Some important differences should be pointed out, though.

[2]The same formula appeared earlier in [7], but in a different context.

[3]This argument was also presented in [9], but only for automorphism modular invariants.

The authors of [8] make two additional assumptions which we want to avoid. First of all, instead of (2.1) they used an orthogonality relation $\sum_{i,\alpha}[R_{(i,\alpha)a}(R_{(i,\alpha)b})^*] = \delta_{ab}$, which follows from an assumed boundary orthogonality condition. Secondly, they postulate some complex conjugation properties of R, namely $(R_{(i,\alpha)a})^* = R_{(i,\alpha)a^c} = R_{(i^*,\alpha^*)a}$, which requires some unspecified Ishibashi and degeneracy label involutions i^* and α^*. No arguments are given why these conditions should hold; indeed, in a general basis in "degeneracy space" these conditions do not hold. We believe it is plausible that a basis exists where they do hold, but fortunately we do not have to deal with this problem, since we never need the complex conjugate of boundary coefficients. We have avoided the boundary orthogonality assumption by imposing a target space string theory condition, namely that sensible Chan-Paton gauge groups should emerge. This is certainly necessary, but of course it would be preferable to derive this from a physical principle on the world sheet.

It should be emphasized that completeness for boundaries is not on equal footing with completeness for bulk operators, i.e. modular invariance, as a consistency condition. Whereas a violation of the latter leads to clearly identifiable inconsistencies in string theory, there are, generically, no obvious inconsistencies associated with violating completeness for boundaries. Indeed, in string theory boundaries are counted with Chan-Paton multiplicities, and no principle is violated if some of these multiplicities vanish (as is often required by tadpole cancellation). On the other hand it is known from many examples that completeness of boundaries corresponds correctly to completeness of the set of branes, as can be verified through dualities. Furthermore in conformal field theory complete sets of boundaries have been found in many cases including the large class of simple current modular invariants [4]. In all well-studied cases completeness emerges as a statement regarding the complete set of 1-dimensional representations of a commutative algebra (the fusion algebra in the "Cardy case" [1], more general classifying algebras[4] in other cases.) Presumably the correct mathematical setting to deal with the general case is still missing, but on the basis of current experience it seems reasonable to assume completeness as a consistency condition for boundary CFT.

Formula (2.3) is not only a consequence of completeness, but, under a very mild assumption, equivalent to it. First of all we start with the definition of the annulus coefficients and derive from it

$$\sum_j A^j{}_{ab} S_{jm} = \sum_\alpha \frac{R_{(m,\alpha)a} R_{(m,\alpha)b}}{S_{0m}} \,. \tag{2.4}$$

We multiply both sides with $A^\ell{}_{b^c c}$ and sum over b:

$$\sum_{j,b} A^j{}_{ab} A^\ell{}_{b^c c} S_{jm} = \sum_{\alpha,b} \frac{R_{(m,\alpha)a} R_{(m,\alpha)b}}{S_{0i}} A^\ell{}_{b^c c} \,. \tag{2.5}$$

[4]Such an algebra for boundary coefficients was first derived in [6] for non-diagonal SU(2) WZW models. This was generalized to simple current automorphisms in any RCFT in [10], who also introduced the name and emphasized the role of these algebras in determining the boundary coefficients. In some cases these algebras are isomorphic to "Pasquier algebras" [11], which were first introduced in [12] for a rather different purpose, namely the study of ADE lattice models.

Consider the left-hand side. Using (2.3) and the Verlinde formula and finally once again (2.4) we get

$$\sum_{j,b} A^j{}_{ab} A^\ell{}_{b^c c} S_{jm} = \frac{S^\ell_m}{S_{0m}} \sum_\alpha \frac{R_{(m,\alpha)a} R_{(m,\alpha)c}}{S_{0m}}\,.$$

Combining this with the right-hand side we obtain

$$\sum_\alpha R_{(m,\alpha)a} \left[\frac{S^\ell_m}{S_{0m}} R_{(m,\alpha)c} - \sum_b R_{(m,\alpha)b} A^\ell{}_{b^c c} \right].$$

Note that there is no summation on m here! For fixed ℓ, m and c we find here a set of conditions of the form

$$\sum_\alpha X_\alpha(m,\ell,c) V^a_\alpha(m) = 0\,,$$

where V stands for R. We have such a condition for each a. The condition says that the vector X_α must be orthogonal to the set of vectors V^a_α, where a runs over the set of boundaries. If the vectors V^a_α, considering all a, span the degeneracy space of m, then this set equations implies that $X = 0$. If on the other hand the equations do not imply $X = 0$ (for some m, ℓ and c), then there must be at least one direction in the degeneracy space of m that is orthogonal to all V^a_α. We can then make a rotation in the degeneracy space of m so that this orthogonal direction coincides, for example, with $\alpha = 0$. Then $V^a_0 = 0$ for all a, or in other words $R_{(m,0)a} = 0$ for all a, so that one Ishibashi label does not couple to any boundary.

Conversely, if we assume that all Ishibashi labels couple to at least one boundary, we find that $X = 0$ (Here "all Ishibashi labels" means "there is no basis in the degeneracy space such that one Ishibashi label completely decouples").

The condition $X = 0$ reads

$$\frac{S^\ell_m}{S_{0m}} R_{(m\alpha)c} - \sum_b R_{(m,\alpha)b} A^\ell{}_{b^c c} = 0\,.$$

Now we multiply by $S_{\ell n}$ and sum over ℓ:

$$\begin{aligned} \frac{R_{(n,\alpha)c}}{S_{0n}} \delta_{mn} &= \sum_b R_{(m,\alpha)b} \sum_\ell S_{\ell n} A^\ell{}_{b^c c} \\ &= \sum_b R_{(m,\alpha)b} \sum_\beta \frac{R_{(n,\beta)b^c} R_{(n,\beta)c}}{S_{0n}}\,. \end{aligned} \tag{2.6}$$

This can be written as

$$\sum_\beta R_{(n,\beta)c} \left[\delta_{\alpha\beta}\delta_{mn} - \sum_b R_{(m,\alpha)b} R_{(n,\beta)b^c} \right].$$

Exactly the same "non-decoupling" assumption regarding Ishibashi labels now yields (2.2).

This suggests a different approach to completeness of boundaries. Instead of postulating it, one could first argue for the validity of (2.3) and then derive completeness from it. The completeness condition (2.3) has a suggestive interpretation in terms of counting

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of boundary couplings. Since A^i_{ab} counts the number of boundary operators Ψ^i_{ab} (open string vertex operators creating representation i with left boundary a and right boundary b), the left and right-hand sides of (2.3) correspond to two ways of counting the coupling of open strings i and j to a boundary: one by first fusing i and j, and one by coupling both directly to the boundary. In the latter case the number of couplings is a sum over all allowed intermediate boundaries c. At this point nothing requires those boundaries to form a complete set. But then the previous argument shows that the counting rule (2.3) can only hold if they *do* form a complete set.

3. Polynomial equations

From (2.2) one easily derives the following polynomial equations for the one-loop open string amplitudes (here $Y_{ijk} = \sum_n S_{in}P_{jn}P_{kn}/S_{0n}$ are integers [13])

$$\sum_b A_{ia}{}^b A_{jb}{}^c = \sum_k N_{ij}{}^k A_{ka}{}^c \,, \tag{3.1}$$

$$\sum_b A_{iab} M^b_j = \sum_l Y_{ij}{}^l M_{la} \,, \tag{3.2}$$

$$\sum_a M^a_i M_{ja} = \sum_l Y^l{}_{ij} K_l \,. \tag{3.3}$$

There are two more equations that can only be derived if there are no degeneracies in Z_{mm^c}. Unlike the previous equations they involve only summations over bulk labels.

$$\begin{aligned} \sum_i A_{iab} A^i_{cd} &= \sum_i A_{iac} A^i_{bd} \,, \\ \sum_i M_{ia} M^i_b &= \sum_i A_{iab} K^i \,. \end{aligned} \tag{3.4}$$

Although in particular the first of these has a nice duality-like graphical interpretation, it does not hold in general, and is in fact explicitly violated by some of the cases discussed in [4].

Equations (3.1), (3.2) and (3.3) can be used in attempts to determine boundary and crosscap coefficients in rational CFT's with exceptional modular invariants and/or non-standard Klein bottle choices.

4. Trace identity

From the definition of the annulus and (2.2) we derive immediately

$$\sum_b A_{ib}{}^b = \sum_{\ell,\alpha} \frac{S_{i\ell}}{S_{0\ell}} \delta_{\alpha\alpha} \tag{4.1}$$

The sum over α is equal to $Z_{\ell\ell^c}$. This may be written as

$$Z_{\ell\ell^c} = (ZC)_{\ell\ell} = (ZSS)_{\ell\ell} = (SZS)_{\ell\ell} \,,$$

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where in the last step modular invariance was used. The two factors S combine with those in (4.1) to yield a fusion coefficient. The final result is

$$\sum_b A_{ib}{}^b = \sum_{j,k} N_i{}^{jk} Z_{jk} \,. \tag{4.2}$$

Note that (4.1) is an interesting and non-trivial test for the C-diagonal part of potential modular invariants. This is independent of the existence of a boundary CFT, since in the form (4.2) the right-hand side is manifestly integer.

This trace identity can be extended to higher order using (2.3), and in general one gets

$$\mathrm{Tr}(A_{i_1} A_{i_2} \cdots A_{i_n}) = \mathrm{Tr}(N_{i_1} N_{i_2} \cdots N_{i_n} Z) \,,$$

where all traces and matrix multiplications are in terms of implicit raised and lowered indices.

Another immediate consequence of the trace identity (4.1) is that

$$\mathcal{N}^{(g)}_{i_1 i_2 \cdots i_n} = \sum_p \frac{S_{i_1 p}}{S_{0p}} \frac{S_{i_2 p}}{S_{0p}} \cdots \frac{S_{i_n p}}{S_{0p}} \frac{1}{(S_{0p})^{2(g-1)}} Z_{pp^c} \tag{4.3}$$

are non-negative integers for any integer $g \geq 1$. Indeed, introducing the standard higher genus Verlinde coefficients

$$N^{(g)}_{i_1 i_2 \cdots i_n} = \sum_p \frac{S_{i_1 p}}{S_{0p}} \frac{S_{i_2 p}}{S_{0p}} \cdots \frac{S_{i_n p}}{S_{0p}} \frac{1}{(S_{0p})^{2(g-1)}} \tag{4.4}$$

one easily shows the identity

$$\mathcal{N}^{(g+1)}_{i_1 i_2 \cdots i_n} = \sum_k N^{(g)}_{i_1 i_2 \cdots i_n}{}^k \sum_b A_{kb}{}^b$$

which explains also the constraint $g \geq 1$.

5. Orientation-sensitive trace identities

All of the previous formulas concerned the annulus amplitudes $A_{ia}{}^b$. In the simple current case there are in general for each modular invariant several choices of crosscap coefficients, each with their own complete set of boundary coefficients. It turns out that the amplitudes $A_{ia}{}^b$ are not sensitive to these differences, essentially because the only effect of the orientation-dependent choices is to change the boundary charge conjugation $A^0{}_{ab}$, which drops out in $A_{ia}{}^b$. The quantities $A^i{}_{ab}$, the physically relevant ones in string theory, are however sensitive to these differences.

Let us first compute

$$\sum_a A^0{}_{aa} \,.$$

To compute this trace we make use of the open string partition function integrality condition

$$A^i_{aa} \geq |M^i{}_a| \,, \qquad \text{and} \qquad A^i_{aa} = M^i{}_a \bmod 2 \,.$$

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For $i=0$ the only possibilities for $A^0{}_{aa}$ are 0 or 1, and hence

$$A^0{}_{aa} = \left(M_a^0\right)^2 .$$

We sum this using (3.3) (note that $M_a^0 = M_{a^c}^0$, because both vanish if $a \neq a^c$). Then we get

$$\sum_a A^0{}_{aa} = \sum_\ell Y^\ell{}_{00} K_\ell .$$

This equation has a simple interpretation, in particular if we write it as

$$\frac{1}{2}\left(\sum_a \left(\delta_{aa} + \left(M_a^0\right)^2\right) = \frac{1}{2}\left(\sum_\ell \left(Z_{\ell\ell^c} + Y^\ell{}_{00} K_\ell\right)\right) .\right.$$

The left-hand side is the number of CP gauge groups. The right-hand side is the number of Ishibashi scalars that survive the Klein bottle projection. To see the latter, note that Y^ℓ_{00} is equal to the Frobenius-Schur indicator of primary i [14, 15], which vanishes for complex fields. Therefore if $\ell \neq \ell^c$ Y^ℓ_{00} vanishes, so that these states contribute with a factor 1/2, precisely the reduction of their multiplicity. If the Klein bottle equals the FS indicator and is non zero, then $\ell = \ell^c$. Each such state contributes a factor 1. If the Klein bottle has the opposite sign, the state is projected with a sign opposite the FS-indicator, which implies that the singlet is projected out.

It was tacidly assumed here that the degeneracies are 0 or 1. For higher multiplicities the interpretation is essentially the same. To illustrate that, consider a label ℓ with $Y^\ell_{00} \neq 0$, so that $\ell = \ell^c$, and with $Z_{\ell\ell} = 2$ (this case was also discussed in [16]). In general, K_ℓ must be an integer with $|K_\ell| \leq Z_{\ell\ell}$, and $K_\ell = Z_{\ell\ell}$ mod 2 Hence the possibilities are now $K_\ell = 0$ or $K_\ell = \pm 2Y^\ell_{00}$. The degeneracy implies that in the extended theory the representation ℓ splits into two representations ℓ^+ and ℓ^-. The partition function has an interpretation as either $\chi^{\ell^+}\bar{\chi}^{\ell^+} + \chi^{\ell^-}\bar{\chi}^{\ell^-}$ or $\chi^{\ell^+}\bar{\chi}^{\ell^-} + \chi^{\ell^-}\bar{\chi}^{\ell^+}$; the former is only consistent with $K_\ell = \pm 2Y^\ell_{00}$, the latter only with $K_\ell = 0$ in order for the "mod-2" condition to be satisfied in the extended theory. After resolving the fixed point in this way we may proceed exactly as in the non-degenerate case above: for $K_\ell = 2Y^\ell_{00}$ two singlets are kept, for $K_\ell = -2Y^\ell_{00}$ both are projected out, and for $K_\ell = 0$ the two resolved fields are combined off-diagonally, yielding only a single singlet. For still higher multiplicities one has to consider more cases, but the general idea remains the same.

We may write this identity also as

$$\sum_\ell \sum_{a,\alpha} R_{(\ell,\alpha)a} R_{(\ell,\alpha)a} = \sum_\ell Y^\ell{}_{00} K_\ell . \tag{5.1}$$

Although we have derived this with a summation over ℓ, it turns out that in all cases studied so far this relation holds also *without summation*!

This conjecture can be rewritten in terms of the trace identity

$$\sum_a A^i_{aa} = \sum_\ell \frac{S^{i\ell}}{S_{0\ell}} Y^\ell{}_{00} K_\ell . \tag{5.2}$$

Note that the right-hand side is not manifestly integer. Therefore this relation — if true — implies a powerful constraint on possible Klein bottle choices.

Unfortunately we have been unable to prove this trace-formula in general, but we can give additional support for it in special cases using the classifying algebra, which follows from the sewing constraints. The classifying algebra reads

$$R_{(m,\alpha)a}R_{(n,\beta)a} = \sum_{\ell,\gamma} X_{(m,\alpha)(n,\beta);(\ell,\gamma)} R_{(\ell,\gamma)a} R_{0a} \,, \tag{5.3}$$

where $X_{pq;r}$ are the structure constants, which are symmetric in p and q. Note that 0 does not have a degeneracy. If we make the very plausible assumption that $R_{0a} = R_{0a^c}$ (in any case these quantities have the same sign, see below) we can sum both sides over a. Then we get

$$\begin{aligned} \sum_{a,\alpha} R_{(m,\alpha)a}R_{(m,\alpha)a} &= \sum_{\ell,\alpha,\gamma} X_{(m,\alpha)(m,\alpha);(\ell,\gamma)} \sum_a R_{(\ell,\gamma)a} R_{0a^c} \\ &= \sum_{\alpha} X_{(m,\alpha)(m,\alpha);0} \,. \end{aligned} \tag{5.4}$$

If we set $n = \beta = 0$ in the classifying algebra (5.3) , we obtain

$$R_{(m,\alpha)a}R_{0a} = \sum_{\ell,\gamma} X_{(m,\alpha)0;(\ell,\gamma)} R_{(\ell,\gamma)a} R_{0a} \,. \tag{5.5}$$

Now note that from the expression for the annulus amplitude we may derive

$$R_{0a}R_{0b} = S_{00} \sum_j A_{jab} S_{j0} \,.$$

Therefore (since in unitary CFT's $S_{j0} > 0$) $R_{0a}R_{0a^c} > S_{00}A_{0aa^c}S_{00} = (S_{00})^2 > 0$. Hence all R_{0a} are non vanishing (and have the same sign). So we can divide both sides of (5.5) by R_{0a} and find

$$R_{(m,\alpha)a} = \sum_{\ell,\gamma} X_{(m,\alpha)0;(\ell,\gamma)} R_{(\ell,\gamma)a} \,.$$

This clearly implies

$$X_{(m,\alpha)0;(\ell,\gamma)} = \delta_{m\ell}\delta_{\alpha\gamma} \,.$$

We can also solve for all X's in terms of R. The result is

$$X_{(m,\alpha)(n,\beta);(\ell,\gamma)} = \sum_a \frac{R_{(m,\alpha)a}R_{(n,\beta)a}R_{(\ell,\gamma)a^c}}{R_{0a}} \,.$$

Note that if all the boundaries are self-conjugate, this quantity is symmetric in the three labels. Then

$$X_{(m,\alpha)(m,\alpha);0} = X_{(m,\alpha)0;(m,\alpha)} = 1$$

so that the sum over α in (5.4) just gives Z_{mm^c}. On the other hand, in that case $\sum_\ell Y^\ell_{00} K_\ell$ must take its maximal value, since with self-conjugate boundaries $\sum_a (M^0_a)^2$ equals the number of boundaries, and hence the number of Ishibashi's. Hence we have

$$\sum_i Z_{ii^c} = \sum_a \left(M_a^0\right)^2 = \sum_i Y^i{}_{00} K_i \le \sum_i Z_{ii^c} \,.$$

Since the inequality must saturate, and since it holds for each i separately, we clearly find

$$Y^m{}_{00} K_m = Z_{mm^c} = \sum_a \sum_\alpha R_{(m,\alpha)a} R_{(m,\alpha)a} \,. \tag{5.6}$$

This estabishes the conjecture for real boundaries. Note that in that case $\sum_a A^i{}_{aa} = \sum_a A^i{}_a{}^a$, so that the left-hand sides of (4.1) and (5.2) are identical. Nevertheless the second trace identity (5.2) contains non-trivial information, since it constrains (and in most cases fixes) the Klein bottle coefficients K^i.

A further generalization can also be proved, namely when boundary conjugation is non trivial, but is linked to charge conjugation in the bulk theory as $R_{(\ell,\gamma)a^c} = R_{(\ell^c,\gamma)a}$. This is true for instance in the Cardy case (i.e. $R_{ma} = S_{ma}$), even in complex CFT's. Then we can derive (5.6) for all real labels m. For complex m on the one hand $Y^m_{00} = 0$, whereas on the other hand $X_{(m,\alpha)(m,\alpha);0} = 0$ vanishes because the classifying algebra coefficients vanish whenever the corresponding fusion coefficients $N_{mm}{}^0$ vanish. This gives an easy explanation for the fact that the only Klein bottle choice consistent with the Cardy case is $K^i = Y^i_{00}$.

What remains to be proved is the "non-saturated" case, where some Klein bottle coefficients are not equal to Y^i_{00}. We were unable to extend the foregoing derivation to such cases, but we did verify that the conjecture holds for the class discussed in [4]. In this paper boundary and crosscap coeficients were presented for all simple current modular invariants (multiplied by charge conjugation) and (presumably) all consistent Klein bottle choices for each invariant. Obviously this includes all non-trivial Klein bottle choices for the charge conjugation invariant. A rather lengthy calculation, which we will not present here, shows that indeed (5.2) holds. Another non-trivial test are the results of [17] for $c = 1$ orbifolds. In this case the Klein bottle amplitude is non standard, but (5.2) nevertheless holds.

It appears that an essential ingredient in boundary CFT (by which we mean conformal field theory on surfaces with boundaries *and* crosscaps) is still missing. We clearly need a deeper understanding of the completeness condition; furthermore a derivation of the trace formula (5.2) — if indeed correct — seems to require some additional insight. It appears that the boundary and crosscap data fit tightly together, and that one may be missing an important piece of the puzzle by focussing only on boundary data, as is the case in most of the literature. We hope that the trace formulas and polynomial equations we have derived or conjectured provide a clue towards an underlying structure. In any case, they are already useful for extending the list of explicit solutions to exceptional cases. Needless to say, we encourage explicit checks of our conjecture (5.2), and would very much like to hear about confirmations or counter examples.

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